Momentum Redistribution in a Briefly, Intensely Irradiated, Structural Element

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An unsteady one-dimensional treatment is undertaken of the response of a semi-infinite porous medium (modeled as a viscoelastic material) to a very intense, brief burst of radiation. Of special interest is the redistribution of the momentum of the impulse by a stress wave, so that the momentum is shared over an appreciable thickness of the medium. (By this strategy, the stress level experienced at the far side of a structural element of finite thickness would be reduced.) Here, a similarity solution is found to describe the Lagrangian-position displacement of the material particles for a broad class of uniform media described by a generic, nonlinear stress/rate-of-strain relation. For a specific stress/strain rate relation sensitive primarily to the porosity of the medium, numerical results are obtained to the pertinent two-point boundary-value problem. In particular, the increase in the maximum (i.e., surface particle) displacement and in the breath of the momentum distribution is found for given magnitude of the impulse.

Nomenclature

$F(s)$	= dimensionless diffusive-friction functional, defined in Eq. (1)
$df(s)/ds$	$= F(s)$
M	= total momentum per unit area
m	$= M/(\sigma_0\tau)$
p	= dimensionless parameter in a model for $F(s)$, defined in Eq. (14)
x	= Lagrangian coordinate
T	= total kinetic energy per unit area
t	= time since impulse
V_1	= volume of a hole
V_2	= volume of a hole plus solid assigned to that hole
$y(x,t)$	= Lagrangian position displacement of particle at x at $t=0$
$y_t(x,t)$	= velocity
$y_x(x,t)$	= strain
$y_{xt}(x,t)$	= strain rate, i.e., rate of elongation per unit of original length
ϵ	$= V_1/V_2$
$\eta(x,t)$	= dimensionless independent variable for similarity, defined in Eq. (6)
ρ	= density of original composite medium
$\sigma(x,t)$	= stress
σ_0	= reference stress for quasiviscous response
τ	= reference time for quasiviscous response
$\Psi(\eta)$	= dimensionless dependent variable for similarity, defined in Eq. (7)

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Introduction

WHEN a structural element (e.g., an ablating heat shield) is exposed to a very intense, brief burst of radiation, there is induced a thin stress wave that travels across the element. A long-considered strategy for minimizing the stress level at the far side of the element is one in which an attempt is made to redistribute the momentum in the stress wave so that the momentum is shared as uniformly as possible over the full thickness of the element. One way to do this is to make the structural element porous.

When the incident pulse blows off some of the material to the left of section B-B in Fig. 1a, the rest of that slab moves to the right and easily crushes the solid segments on and near section A-A (Fig. 1b). As those segments get thicker and shorter, the resistance grows; but, the B-B material continues to advance even after section C-C has acquired a substantial momentum. Figure 2 is an exaggerated diagram of the early and late momentum distribution for a conventional solid and Fig. 3 is a diagram, again idealized, of the early and late momentum distribution for a porous medium. The underlying "first-principle" reason the strategy succeeds lies in the fact that, with a larger hole space, a great deal of heat can be generated in a rapidly distorting material; hence, the final configuration is consistent with conservation of total momentum, although the total mechanical energy has not been conserved. Such is not the case for the homogeneous solid.

Wave propagation in a medium described by a nonlinear stress-strain relation is considered, for example, by Courant and Friedrichs.[1] Here, a stress/strain-rate law is apropos because of the impulse nature of the forcing and the rapidly distorting nature of the material.

Formulation

To analyze the response of this configuration, isolate a segment, such as that between A-A and B-B of Fig. 4. Then, assign a stress-deformation-rate law that states

$$\sigma = \sigma_0 \tau F(y_x)\dot{y}_x \tag{1}$$

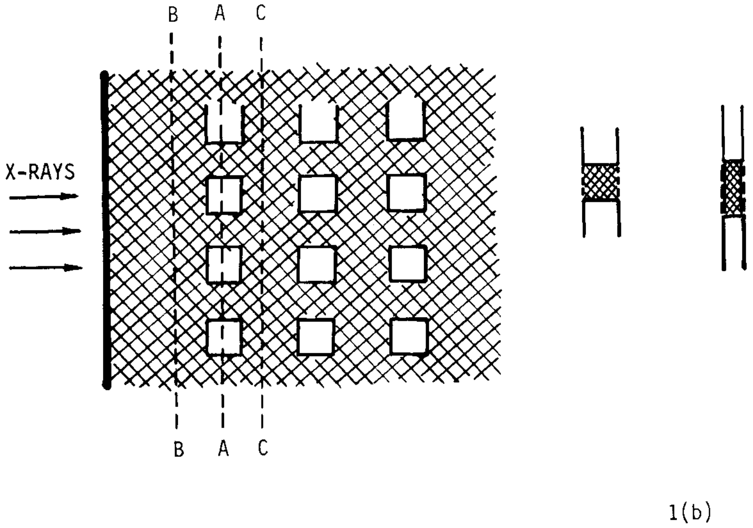

Fig. 1 Effect of porosity on a structural element: a) sections B-B, A-A, and C-C in the porous solid prior to irradiation, with the clear squares indicating the holes and the cross-hatched region the solid material (the density of the original composite medium is ρ); b) after compression following irradiation, the material-filled region (in this close-up of part of section A-A) is shortened and lengthened, such that the material-filled volume is preserved. The hole volume is not preserved (it is reduced) and so the density of the *overall* porous medium is increased; there is appreciable heating association with this appreciable deformation.

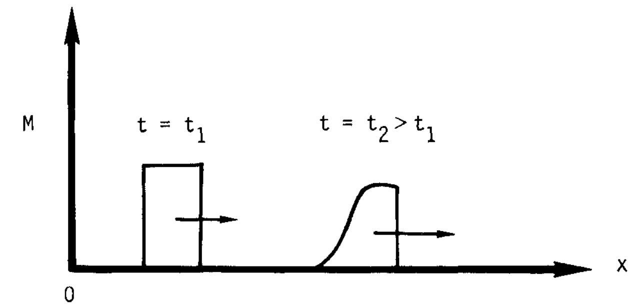

Fig. 2 Total momentum per unit area M vs distance x (where $x=0$ is the surface of the medium on which radiation is incident and $x\to\infty$ is deep into the interior of the medium) for a *conventional solid*. The arrow indicates that the pulse travels into the medium with time; t_1 is the early time after the pulse (at $t=0$) and t_2, the late time after the pulse. The front of the momentum pulse remains sharp as it propagates and the total momentum per unit area is preserved, but the momentum is distributed over more mass.

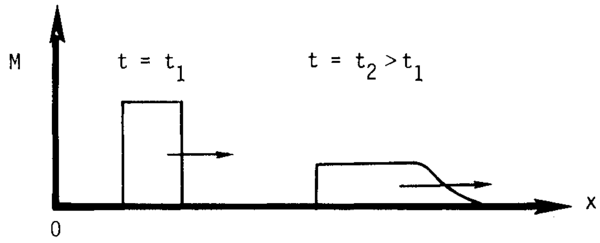

Fig. 3 Same as Fig. 2, but for a porous medium.

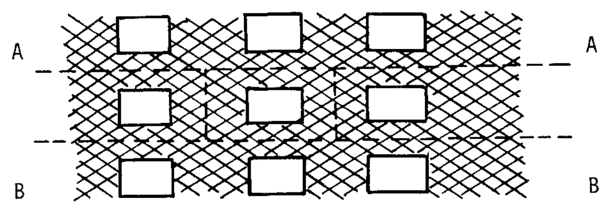

Fig. 4 An isolated segment of the original porous medium.

where $F(y_x)$ is a function that has value unity at $y_x=0$ and that becomes enormous as y_x approaches (V_1/V_2).§ Here, V_1 is the volume of a hole and V_2 is the volume of a hole plus the solid assigned to that hole, as shown in Fig. 4. Also, $y(x,t)$ is the Lagrangian-position *displacement* of a material particle whose original position (at time $t=0$) was at x. The quantity y_x is equal to V_1/V_2 when the hole is closed—hence, the choice of functional behavior of $F(y_x)$ stated above. The dot over the y_x in Eq. (1) denotes partial differentiation with respect to time t and $\sigma_0\tau$ is a reference stress times a reference time that characterizes the plastic (quasiviscous) response of the material. The quantity $\dot{y}_x$ is the rate of elongation per unit of *original* length (*not* current length).

For convenience of manipulation, we define $f(s)$ by

$$f'(s)=F(s) \tag{2}$$

where the prime denotes total derivative. We note that conservation of momentum requires that

$$\rho y_{tt}=\frac{\partial\sigma}{\partial x}=\frac{\partial}{\partial x}\{\sigma_0\tau[f'(y_x)]y_{xt}\} \tag{3}$$

where ρ is the average density of the porous material in its undeformed geometry and partial differentiation with respect to time is now denoted by a subscript. Integration over time t gives

$$\rho y_t=\sigma_0\tau\frac{\partial}{\partial x}[f(y_x)]=\sigma_0\tau[f'(y_x)]y_{xx}=\sigma_0\tau F(y_x)y_{xx} \tag{4}$$

Note that if F is simply unity, Eq. (4) is simply a standard model of linear diffusive friction.¶ The boundary conditions require that the stress and, thus, the strain rate, by Eq. (1), be zero after time $t=0$ at position $x=0$ and, for a very thick solid, that y and y_x and higher derivatives be zero for all time $t\geq 0$ as $x\to\infty$. The description of y_t should be singular at $x=0$ and $t=0$, since the total impulse is of indefinitely large magnitude over an indefinitely small distance, and M, the total momentum per unit area, should be given (at all time $t>0$) by

$$\frac{M}{\rho}=\int_0^\infty y_t\,\mathrm{d}x \tag{5}$$

Note that y_t is the velocity in the Lagrangian formulation and the total velocity over depth is conserved since density is a constant.

§One expects the multiplicative function F in Eq. (1) to be a function of the strain y_x (and of the porosity) because the key property of the porous material in the present context is the change in volume of the solid component. In general, F is a monotonically increasing function of the magnitude of the strain, with F approaching an asymptotically large value as the void volume approaches zero. In this limit, the density of the porous material approaches the density of the solid component.

¶As in many other contexts involving parabolic equations, the diffusive equation (4) describes a wave-like phenomenon. It may be pertinent to note that wave-type phenomena are not limited to hyperbolic systems; for example, the description of waves in shallow water entails an elliptic equation—explicitly, Laplace's equation.

If we let

$$\eta = x/2(\sigma_0 \tau t/\rho)^{1/2} \tag{6}$$

and if we note that $\sigma_0\tau/\rho$ has the usual (length)^2/time dimensions of a diffusion coefficient and if we anticipate that

$$y = 2(\sigma_0 \tau t/\rho)^{1/2}\psi(\eta) \tag{7}$$

then Eq. (4) reduces to

$$F(\psi')\psi'' = 2(\psi - \eta\psi') \tag{8}$$

The form of y chosen can be consistent with $\psi(\infty)=0$ and with the vanishing at $x=0$ of

$$y_{xt}(\eta,t) = -(\eta/2t)\psi_{\eta\eta} \tag{9}$$

Equation (5) then becomes

$$M = 2\sigma_0\tau \int_0^\infty (\psi - \eta\psi')\mathrm{d}\eta \tag{10}$$

Thus, solutions of Eq. (8) must be found for which Eq. (5) is consistent with the postulated momentum and with $\psi(\infty)=0$. The operable combination of factors is $m = M \div (\sigma_0\tau)$.

Note that T, the total kinetic energy per unit area, is given by

$$T = (\sigma_0\tau)^{3/2}(\rho t)^{-1/2}\int_0^\infty (\psi - \eta\psi')^2\mathrm{d}\eta \tag{11}$$

and that it decreases with increasing time as $t^{-1/2}$. The decrease of the total mechanical *energy* per unit area in time implies a broadening of the width of the pulse (i.e., the total *momentum* per unit area is constant in magnitude, but is distributed over more mass).

Some trials with the functional forms of F that depend on depth x (equivalent to tailoring the porosity) could be used to optimize this momentum spreading, but the analysis would be much more intricate because the similarity solution would no longer be useful.

Analysis

For the linear case $F(\psi')=1$,

$$\psi(\eta) = c_1\eta + c_2[\exp(-\eta^2) - \pi^{1/2}\eta\,\mathrm{erfc}(\eta)] \tag{12}$$

where the constraint $\psi(\infty)=0$ gives $c_1=0$ and Eq. (10) gives

$$c_2 = m/(\pi^{1/2}) \tag{13}$$

Although it is not admissible physically, it may be useful to note that, if $F(\psi')=N$ (a positive constant), then

$$\psi(\eta;m,N) = \psi(N^{-1/2}\eta; N^{-1/2}m, 1) \tag{14}$$

Since in general $F(\psi')$ approaches unity as $\eta\to\infty$, in general $\psi(\eta)$ is expected to decay as $\exp(-\eta^2)$ as $\eta\to\infty$.

Attention here is concentrated on the functional

$$F[\psi'(\eta);\,\epsilon,p] = \left[\frac{\epsilon}{\epsilon+\psi'(\eta)}\right]^p \tag{15}$$

where the real constants $\epsilon(>0)$ and $p(\geq 0)$ are assignable and $\psi'(\eta)\leq 0$ [although $\psi(\eta)\geq 0$]. In fact, it may be useful to note that, under Eq. (12), $\psi(\eta)$ monotonically decreases to zero and $\psi'(\eta)$ monotonically increases (algebraically) to zero as η increases from zero to infinity. For Eq. (15), the solution need be obtained for only one value of ϵ since

$$\psi(\eta;m,p,\epsilon) = \left[\psi\left(\eta;\frac{m}{\epsilon},p,1\right)\right]\Big/\epsilon \tag{16}$$

By inspection of Eq. (15), $\epsilon > |\psi'(0)|$. In the next paragraph, it is shown that a relationship exists between $\psi'(0)$ and m for any plausible $F[\psi'(\eta);\ldots]$. Thus, the bound on the quantity $\psi'(0)$ implies a bound on the normalized impulse m.

If Eq. (2) is written in different notation,

$$\frac{\mathrm{d}f[\psi'(\eta)]}{\mathrm{d}\psi'(\eta)} = F[\psi'(\eta)] \tag{17}$$

then, from Eqs. (8) and (17),

$$\frac{\mathrm{d}f[\psi'(\eta)]}{\mathrm{d}\eta} = \frac{\mathrm{d}f[\psi'(\eta)]}{\mathrm{d}\psi'(\eta)}\,\frac{\mathrm{d}^2\psi(\eta)}{\mathrm{d}\eta^2} = 2\left[\psi(\eta) - \eta\frac{\mathrm{d}\psi(\eta)}{\mathrm{d}\eta}\right] \tag{18}$$

$$f[\psi'(\eta)] - f[\psi'(0)] = 2\int_0^\eta [\psi(\eta_1) - \eta_1\psi'(\eta_1)]\mathrm{d}\eta_1 \tag{19}$$

For $\eta\to\infty$, since $\psi'(\infty)=0$, by Eq. (10),

$$f[\psi'(0)] = f(0) - m \tag{20}$$

or, by introduction of the inverse function,

$$\psi'(0) = f^{-1}[f(0) - m] \tag{21}$$

Thus, in solution of the boundary-value problem posed by Eqs. (8) and (10) and the requirement that $\psi\to\exp(-\eta^2)$ as $\eta\to\infty$, one may choose $\psi'(0)$ to ensure consistency with Eq. (10) and may iterate on $\psi(0)$ to achieve consistency of solutions of Eq. (8) with the constraint at $\eta\to\infty$. Briefly, numerical treatment via a "shooting" method entails a single degree of freedom.

For $F=1$, by Eq. (17), $f[\psi']=\psi'$, $f(0)=0$, and $\psi'=f^{-1}[\psi']$; thus, by Eq. (21), $\psi'(0)=-m$, a result confirmed by Eqs. (12) and (13).

For the particular functional of interest here, Eq. (15) becomes

$$f[\psi'] = \epsilon\ln(\epsilon+\psi'),\quad p=1$$

$$f[\psi'] = \frac{\epsilon^p}{1-p}(\epsilon+\psi')^{1-p},\quad p\neq 1 \tag{22}$$

$$f(0) = \epsilon\ln\epsilon,\quad p=1$$

$$f(0) = \epsilon/(1-p),\quad p\neq 1 \tag{23}$$

$$\psi' = \exp\left\{\frac{f[\psi']}{\epsilon}\right\} - \epsilon,\quad p=1$$

$$\psi' = \left\{\frac{(1-p)f[\psi']}{\epsilon^p}\right\}^{1/(1-p)} - \epsilon,\quad p\neq 1 \tag{24}$$

Hence,

$$f^{-1}[\psi'] = \exp\left[\frac{\psi'}{\epsilon}\right] - \epsilon,\quad p=1$$

$$f^{-1}[\psi'] = \left[\frac{(1-p)\psi'}{\epsilon^p}\right]^{1/(1-p)} - \epsilon,\quad p\neq 1 \tag{25}$$

From Eqs. (21), (23), and (25), for $p=1$,

$$\psi'(0) = f^{-1}[\epsilon \ell n \epsilon - m] = \epsilon[\exp(-m/\epsilon) - 1] \tag{26}$$

and for $p \neq 1$,

$$\psi'(0) = f^{-1}\left[\frac{\epsilon}{1-p} - m\right] = \epsilon\left\{\left[1 - \frac{m(1-p)}{\epsilon}\right]^{1/(1-p)} - 1\right\} \tag{27}$$

The appearance of the premultiplier ϵ and of the quotient m/ϵ is consistent with Eq. (16). Thus, for $p<1$, $(1-p)^{-1} > (m/\epsilon)$ to ensure $\psi'(0)<0$; for $p \geq 1$, no constraint arises other than the requirement that $(m/\epsilon)>0$.

Numerical Results

The numerical solution to Eqs. (8) and (10), with $\psi(\infty)=0$, with Eq. (15), in light of Eqs. (16), (26), and (27), is reported here for $\epsilon = 0.25$. The results for the normalized momentum $m = 0.20$ for a range of values of the parameter p are presented in Table 1, and the results for $p=2$ for a range of values of the parameter m, in Table 2. Recall that $\psi'(0)$ is implied by the choice of p and m and that $\psi(0)$ is sought. Numerical experimentation suggests that the error in listed values of $\psi(0)$ is about $\pm 2 \times 10^{-6}$, that the error in the value of η for which the integral for m achieves 90% of its ultimate value is at least ± 0.15, and that the error in the value of η for which the integral for m achieves virtually 100% of its value is about ± 0.15. These error bounds suggest that one must establish $\psi(0)$ virtually to six decimal places before a linear interpolative scheme is useful [implying new trials for $\psi(0)$ from the variances of $\psi(\infty)$ from zero in previous trials].

For the nonlinear diffusive-friction model of Eq. (15), the following results are obtained. For a fixed finite time, the displacement of the surface particle (the maximum displacement for any particle) is decreased inversely as ϵ is increased, for fixed p, *provided* that the normalized momentum m is increased by a factor ϵ; it is recalled that ϵ is the ratio of the volume of a hole to the volume of hole plus solid assigned to it, so ϵ is the porosity. The spatial scale of the momentum distribution is unaltered. For fixed m and ϵ, increasing the factor p ("increasing the nonlinearity of the diffusive friction model") reduces the surface particle displacement moderately, with the spatial scale of momentum distribution increasing by a smaller factor. For fixed p and ϵ, increasing m increases the surface displacement rather notably, with the spatial distribution of momentum spreading more modestly.

Table 1 Material response, for fixed normalized momentum m[a] as a function of the power-law parameter p

p	$\psi(0)$	$-\psi'(0)$	$\eta_{0.9}$[b]	η_∞[c]
0	0.112838	0.2	1.15	2.90
1/2	0.104294	0.16	1.20	2.95
1	0.098438	0.137668	1.25	2.9
2	0.090263	0.111111	1.50	2.95
10	0.0648623	0.0521185	1.65	3.35
100	0.0309506	0.0108304	3.15	4.55

[a] $m=0.20$, $\epsilon=0.25$. [b] $0.9m = 2\int_0^{\eta_{0.9}} (\psi - \eta_1 \psi')\,d\eta_1$. [c] $m \pm 2\int_0^{\eta_\infty} (\psi - \eta_1\psi')\,d\eta_1$.

Table 2 Material response as a function of normalized momentum m for fixed power-law parameter p[a]

p	$\psi(0)$	$-\psi'(0)$	$\eta_{0.9}$[b]	η_∞[b]
0.2	0.090263	0.111111	1.5	2.95
0.3	0.124929	0.136364	1.35	3.10
0.4	0.155543	0.153846	1.45	3.25
0.5	0.183154	0.166667	1.50	3.20
1.0	0.293807	0.2	1.75	3.55
2.0	0.450449	0.222222	2.15	4.00
5.0	0.755082	0.238095	3.10	4.70

[a] $p=2$, $\epsilon=0.25$. [b] See Table 1 for definition.

Future Investigations

Further investigation, to be reported separately, will consider the previously mentioned case of spatially variable porosity $\epsilon(x)$. Since similarity no longer holds in this case, numerical integration of a nonlinear parabolic partial differential equation is required; of course, the invariance in time of the quantity m, the total momentum per area, still holds. Among the examples to be considered are 1) a monotonic porosity whose (finite) value at $x=0$ is decreased relative to a certain constant value, but whose asymptotic (finite) value at $x \to \infty$ is increased relative to that constant value; and 2) a corresponding porosity that monotonically decreases from an enhanced value at $x=0$ to a finite asymptotic value. Variable-porosity cases may be of interest *if* the existence of increased near-surface porosity implies increased local permeability of the porous medium, although, of course, the two properties need not be so related. Increased permeability well may imply increased absorption of ambient water, which undergoes a phase transition to expansive steam at a temperature probably well below the temperature characterizing the phase transition of the solid component of the porous medium. Hence, increased near-surface permeability could imply an increased blow-off of the material under irradiation, especially if the solid component of the porous medium appreciably weakens with the increase in temperature. In such instances, there would be an increased impulse. On the other hand, increased porosity implies reduced peak stress as the disturbance from the impulse propagates through the material. Thus, for a slab of porous material, integrity might be enhanced by smaller near-surface porosity (to inhibit water absorption) and larger porosity in the interior (to reduce the peak stress of a propagating wave). However, the verification of these conjectures awaits further analysis.

Reference

[1]Courant, R. and Friedrichs, K. O., *Supersonic Flow and Shock Waves*, 1st ed., Interscience, New York, 1948, pp. 235-246.